§9. 高斯第二定理(即高斯绝妙定理)

高斯绝妙定理:

曲面的高斯曲率只依赖于第一基本型(而不依赖于第二基本型).

即:对曲面 $r=r(p,q)$,有

$$K=\frac{\det \mathrm{II}}{\det \mathrm{I}}$$

但高斯进一步证明如下公式:

$$\begin{aligned}4(EG-F^2)^2K=&E\left(\frac{\partial E}{\partial q}\cdot\frac{\partial G}{\partial q}-2\frac{\partial F}{\partial p}\cdot\frac{\partial G}{\partial q}+\left(\frac{\partial G}{\partial p}\right)^2\right)\\&+F\left(\frac{\partial E}{\partial p}\cdot\frac{\partial G}{\partial q}-\frac{\partial E}{\partial q}\cdot\frac{\partial G}{\partial p}-2\frac{\partial E}{\partial q}\cdot\frac{\partial F}{\partial q}+4\frac{\partial F}{\partial p}\cdot\frac{\partial F}{\partial q}\right.\\&\qquad\left.-2\frac{\partial F}{\partial p}\cdot\frac{\partial G}{\partial p}\right)\\&+G\left(\frac{\partial E}{\partial q}\cdot\frac{\partial G}{\partial p}-2\frac{\partial E}{\partial p}\cdot\frac{\partial F}{\partial q}+\left(\frac{\partial E}{\partial q}\right)^2\right)\\&-2(EG-F^2)\left(\frac{\partial^2E}{\partial q^2}-2\frac{\partial^2F}{\partial p\partial q}+\frac{\partial^2G}{\partial p^2}\right)\end{aligned}$$

特殊情形: $z=f(x,y)$

$$E=1+f_x^2,\quad F=f_xf_y,\quad G=1+f_y^2,\quad \det(\mathrm{II})=f_{xx}f_{yy}-f_{xy}^2$$

$$\begin{cases} E_x = 2f_x f_{xx} \\ E_y = 2f_x f_{xy} \end{cases} \quad \begin{cases} G_x = 2f_y f_{xy} \\ G_y = 2f_y f_{yy} \end{cases} \quad \begin{cases} F_x = f_{xx} f_y + f_x f_{xy} \\ F_y = f_{xy} f_y + f_x f_{yy} \end{cases}$$

则有 $(E_x G_y - E_y G_x) = 4(f_{xx} f_{yy} - f_{xy}^2)$

$$= 4F \cdot (\det \mathrm{II})$$

故 $K = \dfrac{\det \mathrm{II}}{\det \mathrm{I}} = \dfrac{1}{4F} \cdot \dfrac{E_x G_y - E_y G_x}{(EG - F^2)}$ (若 $F \neq 0$)

如果 $F=0$ 怎么办？ $(f_{xx} f_{yy} - f_{xy}^2) = (f_y f_{xx})_y - (f_y f_{xy})_x$

我们来看高斯论文中的计算： $= (F_x - \frac{1}{2} E_y)_y - (\frac{1}{2} G_x)_x$

$= \frac{1}{2} E_{yy} + F_{xy} - \frac{1}{2} G_{xx}$

记号： $r = r(p,q) = (x(p,q), y(p,q), z(p,q))$

$$\begin{cases} a = x_p, \quad b = y_p, \quad c = z_p \\ a' = x_q, \quad b' = y_q, \quad c' = z_q \end{cases}$$

$$\begin{cases} \alpha = x_{pp}, \quad \beta = y_{pp}, \quad \gamma = z_{pp} \\ \alpha' = x_{pq}, \quad \beta' = y_{pq}, \quad \gamma' = z_{pq} \\ \alpha'' = x_{qq}, \quad \beta'' = y_{qq}, \quad \gamma'' = z_{qq} \end{cases}$$

$$A = bc' - cb', \quad B = ca' - ac', \quad C = ab' - ba'$$

$$\begin{cases} D = A\alpha + B\beta + C\gamma & (1) \\ D' = A\alpha' + B\beta' + C\gamma' & (2) \\ D'' = A\alpha'' + B\beta'' + C\gamma'' & (3) \end{cases}$$

$$\begin{cases} m = a\alpha + b\beta + c\gamma & (4) \\ m' = a\alpha' + b\beta' + c\gamma' & (5) \\ m'' = a\alpha'' + b\beta'' + c\gamma'' & (6) \end{cases}$$

$$\begin{cases} n = a'\alpha + b'\beta + c'\gamma & (7) \\ n' = a'\alpha' + b'\beta' + c'\gamma' & (8) \\ n'' = a'\alpha'' + b'\beta'' + c'\gamma'' & (9) \end{cases}$$

$$\Delta = EG - F^2 = (a^2+b^2+c^2)(a'^2+b'^2+c'^2) - (aa'+bb'+cc')^2$$

$$= A^2 + B^2 + C^2$$

Step 1: $K = \dfrac{DD'' - D'^2}{(A^2+B^2+C^2)^2}$

证明：注意到

$|r_p \times r_q|^2 = EG - F^2$，故有

$$\langle r_{pp}, r_p \times r_q \rangle \langle r_{qq}, r_p \times r_q \rangle - \langle r_{pq}, r_p \times r_q \rangle^2$$

$$= DD'' - D'^2$$

注意到

$$r_p \times r_2 = \begin{vmatrix} i & j & k \\ a & b & c \\ a' & b' & c' \end{vmatrix} = A \cdot i + B j + C \cdot k$$

Thus 上式是显然的.

Step 2: (i) (1), (4), (7)

$$\underbrace{\begin{pmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{pmatrix}}_{T} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} D \\ m \\ n \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} A & b'C-Bc' & Bc-bC \\ B & Ac'-a'C & aC-Ac \\ C & Ba'-b'A & Ab-aB \end{pmatrix}}_{T''^{*}} \begin{pmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = T^{*} \begin{pmatrix} D \\ m \\ n \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \det T \overset{\Delta}{} & & \\ & \det T'' \overset{\Delta}{} & \\ & & \det T' \overset{\Delta}{} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} AD - a(nF-mG) \mp a'(mF-nE) \\ BD - b(nF-mG) \mp b'(mF-nE) \\ CD - c(nF-mG) - c'(mF-nE) \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} (AD = \Delta\alpha + \quad\quad) \times \alpha'' \\ (BD = \Delta\alpha + \quad\quad) \times \beta'' \\ (CD = \Delta\alpha + \quad\quad) \times \gamma'' \end{array}\right\} +$$

$$\Rightarrow D \cdot D'' = (\alpha\alpha'' + \beta\beta'' + \gamma\gamma'')\Delta + m''(nF-mG) + n''(mF-nE) \quad (10)$$

(ii) (2), (5), (8) $\Rightarrow$

$$(D')^2 = (\alpha'^2 + \beta'^2 + \gamma'^2)\Delta + m'(n'F-m'G) + n'(m'F-n'E) \quad (11)$$

(10),(11)
$\Rightarrow$

$$DD'' - D'^2 = (\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2)\Delta$$
$$+ E(n'^2 - nn'') + F(nm'' - 2m'n' + mn'') + G(m'^2 - mm'') \qquad (12)$$

(iii)
$$\left.\begin{array}{l} E_p = 2m \;;\; E_q = 2m' \;; \\ F_p = m' + n \;;\; F_q = m'' + n' \\ G_p = 2n' \;;\; G_q = 2n'' \end{array}\right\} \Rightarrow$$

$$\begin{cases} m = \frac{1}{2}E_p \;;\; m' = \frac{1}{2}E_q \;;\; m'' = F_q - \frac{1}{2}G_p \\ n = F_p - \frac{1}{2}E_q \;;\; n' = \frac{1}{2}G_p \;;\; n'' = \frac{1}{2}G_q \end{cases} \qquad (13)$$

(iv) $\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - (\alpha'^2 + \beta'^2 + \gamma'^2) = \underline{(\alpha\alpha'' - \alpha'^2)} + \underline{(\beta\beta'' - \beta'^2)} + \underline{(\gamma\gamma'' - \gamma'^2)}$

情形与图4类似!

$$= n_q - n'_p = m''_p - m'_q$$

$$= -\frac{1}{2}E_{qq} + F_{pq} - \frac{1}{2}G_{pp} \qquad (14)$$

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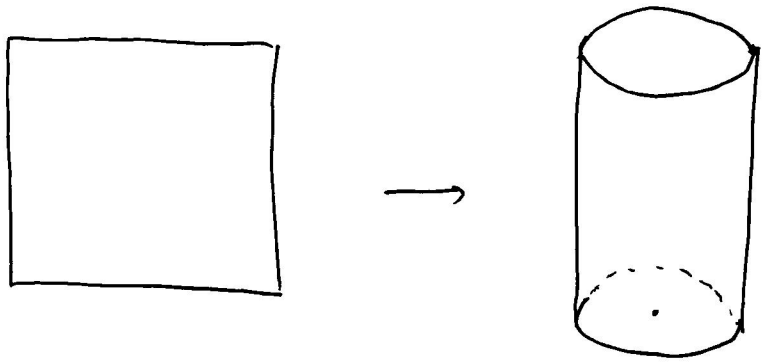
<u>注记</u>: 由于曲面局部都是图(利用到坐标平面的投影), 故对于图的高斯绝妙定理的证明推出一般曲面的高斯绝妙定理.

定义 1（等距参数）

设 $S, \tilde{S} \subset \mathbb{R}^3$ 为曲面，$\varphi: S \to \tilde{S}$ 为一双射，φ 称为等距参数，若曲面 S 上的任意曲线 C 与 C 在 $\tilde{S}$ 对应的曲线 $\tilde{C}$（$=\varphi(C)$）长度相等。

例子：



$$(u, v, 0) \longmapsto (\cos u, \sin u, v)$$

注意到这并非由 $\mathbb{R}^2$ 中合同变换诱导。

命题：设曲面 S 和 $\tilde{S}$ 的参数别为 $r = r(u, v)$ 和 $\tilde{r} = (\tilde{u}, \tilde{v})$。

$\varphi: (u, v) \longmapsto (\tilde{u}, \tilde{v})$ 为双射（光滑）则 φ 是等距参数

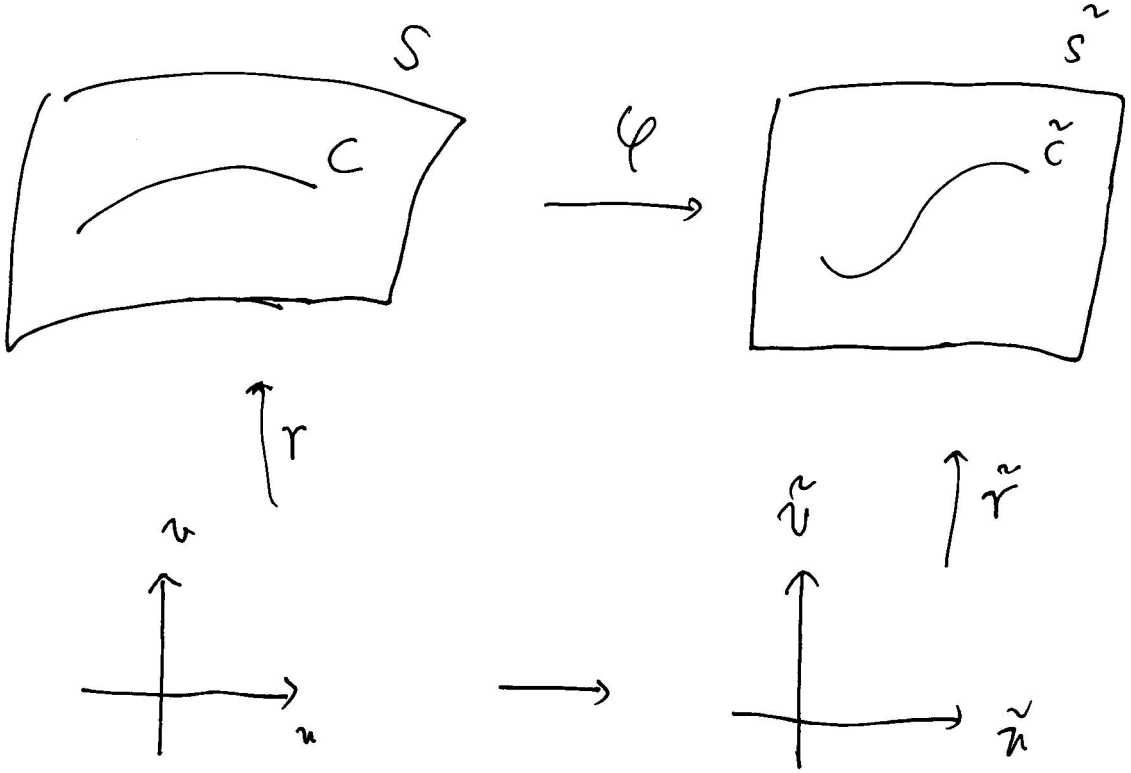
当且仅当

$$\begin{bmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{bmatrix} = J(\varphi) \begin{bmatrix} \tilde{E}(\tilde{u},\tilde{v}) & \tilde{F}(\tilde{u},\tilde{v}) \\ \tilde{F}(\tilde{u},\tilde{v}) & \tilde{G}(\tilde{u},\tilde{v}) \end{bmatrix} J(\varphi)^T$$

其中 $J(\varphi) = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix}$ 为 φ 的 Jacobi 矩阵。

证明：令 $C = r(u(t), v(t)) \subset S$

则 $\tilde{C} = \varphi(C) = \tilde{r}(\tilde{u}(t), \tilde{v}(t)) \subset \tilde{S}$



则 $l(C) = \int_0^S \left[\left(\frac{du}{dt}, \frac{dv}{dt} \right) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} \right]^{1/2} dt$

C的弧长

$$l(\varphi(C)) = \int_0^S \left[\left(\frac{d\tilde{u}}{dt}, \frac{d\tilde{v}}{dt} \right) \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} \begin{pmatrix} \frac{d\tilde{u}}{dt} \\ \frac{d\tilde{v}}{dt} \end{pmatrix} \right]^{1/2} dt$$

~~Let S→0~~. Then $l(C) = l(\varphi(C))$. Let $S \to 0$

$$\Rightarrow \left(\frac{du}{dt} \ \frac{dv}{dt} \right) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix}$$

$$= \left(\frac{du}{dt} \ \frac{dv}{dt} \right) J(\varphi) \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} J(\varphi)^T \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix}$$

因为 $\left(\frac{du}{dt}, \frac{dv}{dt}\right)$ 任意，所得结论：

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = J(\varphi) \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} J(\varphi)^T$$

反之，弧长微元保持，则弧长是其在定义区间上积分，故任意曲线弧长保持。

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命题：记号如上；φ 是等距变换，当且仅当 φ 保持第一基本型，即

$$\boxed{I(\dot{u}, \dot{v}) = \tilde{I}(\dot{u}, \dot{v})}$$

例：螺旋面

$$r(u,v) = (u\cos v,\ u\sin v,\ v)$$

悬链面

$$r(\rho,\theta) = (\rho\cos\theta,\ \rho\sin\theta,\ \operatorname{arccosh}(\rho))$$

$$I(u,v)_{(du,dv)} = \begin{pmatrix} 1 & 0 \\ 0 & 1+u^2 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$= (du)^2 + (1+u^2)(dv)^2$$

$$\tilde{I}(\rho,\theta) = \frac{\rho^2}{\rho^2-1}(d\rho)^2 + \rho^2(d\theta)^2$$

令 $\varphi: (u,v) \longmapsto (\rho,\theta) = (\sqrt{1+u^2},\ v)$

$$d\rho = \frac{u\,du}{\sqrt{1+u^2}}, \quad d\theta = dv$$

$$\Rightarrow I(\rho,\theta) = \left(\frac{1+u^2}{1+u^2-1}\right)\cdot\frac{u^2\,du^2}{(1+u^2)} + (1+u^2)(dv)^2$$

$$= I(u,v)$$

$$\text{II}(u,v) = \begin{matrix}(du,dv)\end{matrix}\begin{pmatrix} 0 & \frac{-1}{\sqrt{1+u^2}} \\ \frac{-1}{\sqrt{1+u^2}} & 0 \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}; \quad \text{II}(\rho,\theta) = (d\rho,d\theta)\begin{pmatrix} \frac{1}{\rho^2-1} & 0 \\ 0 & \frac{-1}{\rho^2-1} \end{pmatrix}\begin{pmatrix} d\rho \\ d\theta \end{pmatrix}$$

$$\text{II}(\rho,\theta) = \frac{1}{(1+u^2)-1}\,\frac{u^2(dv)^2}{1+u^2} \qquad \frac{1}{\rho^2-1}(d\rho)^2 - \frac{1}{\rho^2-1}(d\theta)^2$$

$$- \frac{1}{(1+u^2)-1}(dv)^2$$

$$= \left(\frac{1}{1+u^2}\right)(du)^2 - \frac{1}{u^2}(dv)^2$$

$$\neq I(u,v)$$

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定理：高斯曲率在等距变换下不变. 即 $\varphi: S \to \tilde{S}$ 等距变换.

则 $\forall p \in S$, 有

$$K(p) = K(\varphi(p)).$$

证明：由高斯绝妙定理知，K由第一基本型唯一确定. 而等距变换保持第一基本型. 定理得证. #

推论：平面与球面之间不存在等距同构变换.

回忆：单位球面在球极坐标系表达为

$$I(\theta,\varphi) = (d\theta)^2 + (\cos^2\theta)(d\varphi)^2$$

上述推论也可表述为

推论'：球面上不存在参数变换，使得其第一基本型在新参数系下表达式为 $(du)^2+(dv)^2$.

证明：设 $\varphi: (u,v)\in U\subset\mathbb{R}^2 \longrightarrow \tilde{U}\subset\mathbb{R}^2$，$\tilde{U} \overset{(\theta,\varphi)}{\shortparallel}$

为参数变换.

我们把 U 看成一个平面片，即 U 上度量为~~标数~~欧氏度量

$(du)^2+(dv)^2$.

则 ~~φ是一个参数变换~~ 我们有关系式：

~~$I(u,v) = J(\varphi) I($~~

$$\begin{pmatrix} E(u,v), & F(u,v) \\ F(u,v), & G(u,v) \end{pmatrix} = J(\varphi)\begin{pmatrix} E(\theta,\varphi), & F(\theta,\varphi) \\ F(\theta,\varphi) & G(\theta,\varphi) \end{pmatrix} J(\varphi)^T$$

故 $$\begin{pmatrix} E(u,v), & F(u,v) \\ F(u,v), & G(u,v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Longleftrightarrow$ φ是等距变换.

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容易验证球面上不存在参数使得其第一基本型为单位

$$I = (du, dv)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}.$$

但是在球极坐标下,

$$I = (du, dv)\begin{pmatrix} \frac{4}{(1+(u^2+v^2)^2)^2} & 0 \\ 0 & \frac{4}{(1+(u^2+v^2)^2)^2} \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

$$= \frac{4}{(1+(u^2+v^2)^2)^2}\left[(du)^2 + (dv)^2\right]$$

这样的坐标系称为等温坐标系, 其严格定义如下:

定义 (保角映射)

$\varphi: S \to \tilde{S}$ 双射. φ 是保角映射, 若 φ 保持任意两相交曲线的夹角.

命题: φ 是保角映射 当且仅当 存在正函数 λ, 使得

$$\boxed{\tilde{I} = \lambda^2 \cdot I.}$$

证明: 令 $C_1, C_2 \subset S$, $P \in C_1 \cap C_2$, C_1 与 C_2 在 P 处正交相交.

即 $T_P C_1 = \mathbb{R}\{\lambda_1 \underbrace{r_u(p) + \mu_1 r_v(p)}_{=: v_1}\}$

$T_P C_2 = \mathbb{R}\{\lambda_2 r_u(p) + \mu_2 \underbrace{r_v(p)}_{=: v_2}\}$

C_1 与 C_2 在 P 点处的夹角指 $\cos(\angle_P C_1 C_2) =$

$$\frac{\langle v_1, v_2 \rangle}{|v_1| \cdot |v_2|}$$

令 φ 是一可微双射（即微同胚）. 则

$\varphi(C_1)$ 与 $\varphi(C_2)$ 在 $\varphi(P)$ 处的夹角指 $\cos(\angle_{\varphi(P)} \varphi(C_1)\varphi(C_2))$

$$\frac{\langle \varphi_*(v_1), \varphi_*(v_2) \rangle}{|\varphi_*(v_1)| \, |\varphi_*(v_2)|}$$

其中: $(\varphi_*)_P : T_P S \longrightarrow T_{\varphi(P)} \tilde{S}$ 为 φ 的切映射

$$\{r_u, r_v\} \longmapsto \{r_{\tilde{u}}, r_{\tilde{v}}\} \cdot J(\varphi)$$

即 $(\varphi_*)_P(r_u) = \frac{\partial \tilde{u}}{\partial u} r_{\tilde{u}} + \frac{\partial \tilde{v}}{\partial u} r_{\tilde{v}}$

$(\varphi_*)(r_v) = \frac{\partial \tilde{u}}{\partial v} r_{\tilde{u}} + \frac{\partial \tilde{v}}{\partial v} r_{\tilde{v}}$

所以 φ 是保角映射 $\Longleftrightarrow$

$$\frac{\langle v_1, v_2 \rangle}{|v_1| \cdot |v_2|} = \frac{\langle \varphi_*(v_1), \varphi_*(v_2) \rangle}{|\varphi_*(v_1)| \cdot |\varphi_*(v_2)|}$$

假定 $\tilde{I} = \lambda^2 \cdot I$.

有 $J(\varphi)\tilde{I}J(\varphi)^T = \lambda^2 I$.

即
$$\begin{pmatrix} \langle \varphi_*(r_u), \varphi_*(r_u)\rangle, \langle \varphi_*(r_u), \varphi_*(r_v)\rangle \\ \langle \varphi_*(r_v), \varphi_*(r_u)\rangle, \langle \varphi_*(r_v), \varphi_*(r_v)\rangle \end{pmatrix} \quad (*)$$

$$= \lambda^2 \begin{pmatrix} \langle r_u, r_u\rangle, \langle r_u, r_v\rangle \\ \langle r_u, r_v\rangle, \langle r_v, r_v\rangle \end{pmatrix}$$

故
$$\frac{\langle \varphi_*(v_1), \varphi_*(v_2)\rangle}{|\varphi_*(v_1)| \cdot |\varphi_*(v_2)|}$$

$$= \frac{(\lambda_1, \mu_1)\begin{pmatrix} \langle \varphi_*(r_u), \varphi_*(r_u)\rangle, \langle \varphi_*(r_u), \varphi_*(r_v)\rangle \\ \langle \varphi_*(r_v), \varphi_*(r_u)\rangle, \langle \varphi_*(r_v), \varphi_*(r_v)\rangle \end{pmatrix}\begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix}}{\left(\sqrt{(\lambda_1, \mu_1)\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix}}\right)\left(\sqrt{(\lambda_2, \mu_2)\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix}}\right)}$$

代入 (*)
$$= \frac{\lambda^2 (\lambda_1, \mu_1)\begin{pmatrix} \langle r_u, r_u\rangle, \langle r_u, r_v\rangle \\ \langle r_v, r_u\rangle, \langle r_v, r_v\rangle \end{pmatrix}\begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix}}{\lambda^2\left(\sqrt{(\lambda_1, \mu_1)\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix}}\right)\left(\sqrt{(\lambda_2, \mu_2)\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix}}\right)}$$

$$= \frac{\langle v_1, v_2 \rangle}{|v_1| \cdot |v_2|}$$

反之，设 φ 为保角变换. 欲证，$\exists \lambda > 0$

$$\tilde{I} = \lambda^2 \cdot I .$$

任取 $p \in S$. 只需证

$$\tilde{I}(p) = (\lambda(p))^2 \cdot I(p)$$

不妨设 $I(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\Longleftrightarrow$ 取 $\{\overset{e_1}{\overset{\shortparallel}{r_u(p)}}, \overset{e_2}{\overset{\shortparallel}{r_v(p)}}\}$ 是 T_pS 的单位正交标架.

这是因为 ~~$\exists J(p)$ 可逆使~~ S 上的参数变换，其 Jacobi 矩阵为 J，使得.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cancel{I(p)} = J(p) I J(p)^T$$

则

$$J(p)\, \tilde{I}(p)\, J^T(p) = \cancel{J(p)\left(\lambda^2(p) \cdot I(p)\right) \cdot J(p)^T}$$

$$= \cancel{\lambda^2(p)\left(J(p)\, J(p)^T\right)}$$

$$= J(p)\left(\lambda^2(p)\, I(p)\right) \cdot J(p)^T$$

$$= \lambda^2(p)\left(J(p)\, I(p)\, J(p)^T\right)$$

$$\shortparallel$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

显然:

$$\langle \varphi_*(e_1), \varphi_*(e_2)\rangle = \left(|\varphi_*(e_1)|\,|\varphi_*(e_2)|\right) \frac{\langle e_1, e_2\rangle}{\underset{\overset{\shortparallel}{1}}{|e_1|}\cdot\underset{\overset{\shortparallel}{1}}{|e_2|}}$$

$$= 0$$

故 $J(\varphi)\,\tilde{I}\,J(\varphi)^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

只需证 $\lambda_1 = \lambda_2$ (>0 显然)

考虑 $\left\{\begin{array}{l} f_1 = \cos\theta\, e_1 + \sin\theta\, e_2 \\ f_2 = -\sin\theta\, e_1 + \cos\theta\, e_2 \end{array}\right\} \in T_pS$

则 $\langle f_1, f_2\rangle = 0 \Rightarrow$

$$0 = \langle \varphi_*(f_1), \varphi_*(f_2)\rangle = (-\cos\theta\cdot\sin\theta)\,|\varphi_*(e_1)|^2 + (\cos\theta\,\sin\theta)\,|\varphi_*(e_2)|^2$$

取 $\theta = \pi/4$. 使得 $\cos\theta\cdot\sin\theta \neq 0$

$$\Rightarrow |\varphi_*(e_1)|^2 = |\varphi_*(e_2)|^2 \quad 即 \quad \lambda_1 = \lambda_2.$$

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推论: 等距变换是保角变换.

~~所以有以下命题~~

推论

命题：球极投影是球面与平面之间的保角变换.

对一般的曲面，我们有深刻的

定理：任意曲面上每一点有一邻域，使得存在参数系 (u,v)，

使得：
$$I = \lambda^2(u,v)\left[(du)^2+(dv)^2\right].$$

即每一点有一邻域可以与平面一区域建立保角变换。

问题：设 S_1, S_2 为被 (u,v) 参数化的曲面。设 K_i 为 S_i 的高斯曲率

且 $K_1(u,v) = K_2(u,v)$, $\forall (u,v)\in U$.

是否存在 S_1 到 S_2 的等距变换？

命题：在等温参数下，高斯曲率表达式为

(即 $I = \lambda^2(du^2+dv^2)$)

$$K = -\frac{4}{\lambda^2}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)(\log\lambda)$$

$$\left[= -\frac{4}{\lambda^2}\partial\delta\log\lambda\right] \qquad \begin{cases}\dfrac{\partial}{\partial z} \overset{\triangle}{=} \dfrac{1}{2}\left(\dfrac{\partial}{\partial u} - i\dfrac{\partial}{\partial v}\right)\\[2mm] \dfrac{\partial}{\partial \bar z} \overset{\triangle}{=} \dfrac{1}{2}\left(\dfrac{\partial}{\partial u} + i\dfrac{\partial}{\partial v}\right)\end{cases}$$

证明：利用高斯的曲率公式代入.

#

命题：设 (u,v) 为曲面 S_i, $i=1,2$ 的等温参数. 则 S_1 与 S_2 等距同构.

$\Longleftrightarrow$ $\lambda_1 = \lambda_2$，其中 $I_i = \lambda_i^2(du^2+dv^2)$ 为 S_i 的第一基本型.

证明：显然.

定义：设 $f(u,v)$ 为区域 $U=\{u,v\}$ 上的可微的函数. 若

满足 $\Delta(f)=0$. 称 f 为调和函数.

命题：设 $z=u+iv$. 若 $f(z)$ 为解析函数，则

$\mathrm{Re}f$, $\mathrm{Im}f$, $|f|^2$ 均为调和函数

证明：可验证.

#

命题：设 $f(u,v)$ 为区域 U 上的正调和函数，则 $I(u,v)=e^{f}(du^2+dv^2)$，高斯曲率为0，但不与平面等距同构.